



# Finitary formal topologies and Stone's representation theorem

Francesco Ciraulo<sup>a</sup>, Giovanni Sambin<sup>b,\*</sup>

<sup>a</sup> Dipartimento di Matematica ed Applicazioni - Università degli Studi di Palermo - Via Archirafi 34, 90123 Palermo, Italy

<sup>b</sup> Dipartimento di Matematica Pura ed Applicata - Università degli Studi di Padova - Via Trieste 63, 35121 Padova, Italy

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## ABSTRACT

We study the concept of finitary formal topology, a point-free version of a topological space with a basis of compact open subsets. The notion of finitary formal topology is defined from the perspective of the Basic Picture (introduced by the second author) and thus it is endowed with a binary positivity relation. As an application, we prove a constructive version of Stone's representation theorem for distributive lattices. We work within the framework of a minimalist foundation (as proposed by Maria Emilia Maietti and the second author). Both inductive and co-inductive methods are used in most proofs.

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## 1. Introduction

Formal topology is the *constructive* counterpart of the standard notion of topological space. Throughout this article, we use the term “constructive” as a synonym of “formalizable in minimal type theory”. Minimal Type Theory (mTT for short), is a foundational theory which has recently been introduced by Maria Emilia Maietti and the second author (see [7]); its peculiarity is to be the common core of other commonly used foundational theories, namely Zermelo-Fraenkel set theory, Martin-Löf's intuitionistic type theory, topos theory, Aczel's CZF. Compatibility with other foundations forces mTT to be a predicative and intuitionistic theory, like Martin-Löf's one, of which it inherits the style, but with a considerable difference about the axiom of choice which, in fact, can be proved in Martin-Löf's theory but cannot in mTT. Hence mTT can briefly be described as Martin-Löf Type Theory deprived of the axiom of choice. This is possible thanks to the fact that, in the framework of mTT, every proposition can be seen as a set, but the converse is not assumed; this is enough to prevent the formal system from proving any choice principle. Note that, because of its relationship with Martin-Löf's theory, several papers devoted to that theory can be referred to mTT as well; in particular, we will be using some concepts and notations introduced in [8] and [13].

From the above discussion, some features of mTT follow:

- a set is defined when we are given rules to introduce and handle its elements (see [8]);
- a subset of a given set is a propositional function with (at most) one free variable ranging over that set (see [13]);
- neither the law of excluded middle nor the powerset axiom nor the axiom of choice is allowed;
- every proof can be seen as an effective method or program; in other words, mTT satisfies the proofs-as-programs paradigm.

From a constructive point of view, classical topological spaces are, in general, intractable objects. As a paradigm, let us look at the real line endowed with its natural topology. One can see that the collection of all classical reals cannot be defined as a set (in mTT, as well as in other constructive approaches); on the contrary, the collection of all open intervals with

\* Corresponding author.

E-mail addresses: [ciraulo@math.unipa.it](mailto:ciraulo@math.unipa.it) (F. Ciraulo), [sambin@math.unipd.it](mailto:sambin@math.unipd.it) (G. Sambin).

URLs: <http://www.math.unipa.it/~ciraulo/> (F. Ciraulo), <http://www.math.unipd.it/~sambin/> (G. Sambin).

rational end-points is a basis for the natural topology of the real line; that basis is a (constructively acceptable) set since it can be identified with  $\mathbb{Q} \times \mathbb{Q}$ . So it is natural to study a topological space from the point of view of a suitable basis of its topology and without mentioning points. This idea, carried out in a predicative foundation, has prompted the definition of Formal Topology (see [11]) and, more recently, of Basic Topology (see [12]). A brief introduction to these notions is given in section 2.

A finitary formal topology is one in which each open subset belonging to the basis is compact. When the first paper on formal topologies [11] appeared in 1987, it already contained a section on finitary topologies, there called Stone topologies. The same topic was then resumed and widened in [10] by Negri in 1996.

In the present paper we are going to refresh the look of finitary topologies; for the most part, this means to adapt them to the new definition of formal topology proposed in [12]: this new definition lacks the unary positivity relation, written  $Pos$ , which in turn is replaced by a binary positivity relation written  $\ltimes$  (see section 2 for definitions). Hence, the main aim of the present work is to study which changes the theory of finitary topologies undergoes because of the introduction of  $\ltimes$ .

Finally, a remark: the present paper is self-contained with regard to definitions; on the contrary it lacks motivations and intuitive explanations which can be found in [12].

## 2. Finite subsets

Before turning our attention to topology, we have to recall briefly some basic notions about subsets introduced in [13]. A subset  $U$  of a given set  $S$  is just a propositional function with at most one free variable, say  $U(x)$ , over that set. The membership relation between an element  $a \in S$  and a subset  $U \subseteq S$  is written  $a \in U$  and its intended meaning is that  $a$  belongs to  $S$  and  $U(a)$  is true. Important examples of subsets are the empty subset, written  $\emptyset$ , the total subset, denoted by  $S$ , and the singletons  $\{a\}$  for  $a \in S$ , which correspond to the propositions  $\perp$  (false),  $\top$  and  $x = a$ , respectively.

Given two subsets, say  $U$  and  $V$ , we say that  $U$  is included in  $V$ , written  $U \subseteq V$ , when  $(\forall x \in S)(U(x) \rightarrow V(x))$  is a true sentence. Of course,  $U = V$  means  $U \subseteq V$  and  $V \subseteq U$ , that is  $(\forall x \in S)(U(x) \leftrightarrow V(x))$ . Hence, equality between subsets is extensional in the sense that two subsets can be equal even if they are defined by different (although equivalent) propositions; intuitively, we can say that two subsets are equal when they share the same elements.

The usual operations on subsets are defined by reflecting the corresponding connectives of intuitionistic logic. Thus  $U \cap V$  is the propositional function  $U(x) \& V(x)$  while  $U \cup V$  is  $U(x) \vee V(x)$ . Infinitary operations are also available, such as the union of a set-indexed family of subsets (details can be found in [13]). Note that an operation corresponding to implication is also definable; in particular, given a subset  $U$ , we denote by  $-U$  the proposition  $\neg U(x) \equiv U(x) \rightarrow \perp$ .

Finally, we use

$$U \text{ } \tilde{\cap} \text{ } V \tag{1}$$

as an abbreviation for  $(\exists x \in S)(U(x) \& V(x))$ . Note that, because of the constructive foundation we are using,  $U \text{ } \tilde{\cap} \text{ } V$  is more informative than  $U \cap V \neq \emptyset$ .

We write  $\mathcal{P}S$  for the collection of all subsets of the set  $S$ . It is surely not a set in the framework of mTT: to assume the powerset axiom breaks compatibility with Martin-Löf type theory (for more details on this, see [7]).

In order to study finitary formal topologies, a certain knowledge of the concept of finite subset is needed. This may appear as a trivial task, but it is not: remember we are working in a very weak foundation, so not all intuitive facts about finiteness can be formally proved in mTT. For this reason, a stock of safe properties seems desirable; in fact, a paper on this is in preparation by the same authors (see [4]). Here we give a very brief and informal introduction to the matter.

Let  $S$  be a set and let  $\{a_1, \dots, a_n\}$  ( $n \geq 0$ ) be a (possibly empty) list of (not necessarily distinct) elements of  $S$ . Then the formula  $K(x) \equiv (x = a_1 \vee \dots \vee x = a_n)$  defines a subset of  $S$ , say  $K$ . A subset defined in this way is what we call a *finite subset*; we write  $K \subseteq_{\omega} U$  to say that  $K$  is a finite subset contained in  $U$ . This notion coincides with that of “finitely indexed” in [15] and that of “finitely enumerable” in [2]. Of course, the empty subset  $\emptyset$  is finite as well as every singleton. It is easily seen that  $\mathcal{P}_{\omega}S$ , the collection of all finite subsets of  $S$ , is closed under binary unions, but not under intersections (unless the equality in  $S$  is effectively decidable). Moreover, enough unpleasantly, a subset of a finite subset is *not* finite, in general. On the other hand, it is always decidable whether a finite subset is empty or not (look at anyone of the lists corresponding to it and decide whether it is the empty list or not).

The collection  $\mathcal{P}_{\omega}S$  is set-indexed by the set  $List(S)$  of all finite lists over  $S$ ; in fact it is possible to define a function, say  $dec$ , from  $List(S)$  to  $\mathcal{P}_{\omega}S$  such that  $dec(\{a_1, \dots, a_n\})$  is the subset  $(x = a_1 \vee \dots \vee x = a_n)$ . This fact allows us to treat  $\mathcal{P}_{\omega}S$  almost as a set. As an example, each quantification over it can be given constructive meaning by formally quantifying over  $List(S)$ . Of course  $\mathcal{P}_{\omega}S$  can be identified with the setoid<sup>1</sup>  $List(S)/\sim$  where  $l_1 \sim l_2$  if  $dec(l_1) \leftrightarrow dec(l_2)$ . Moreover, we can use it to construct new setoids by means of the set-constructors of mTT, such as indexed sum and dependent product. For more details on this see [4].

From now on,  $S$  and  $T$  will denote sets,  $a, b, c, d, \dots, u, v, \dots$  elements of a set and  $U, V, W, \dots$  subsets of a given set.

<sup>1</sup> To our purpose, a setoid is just a set equipped with an equivalence relation.

### 3. Formal topologies and their morphisms

Many of the definitions we are going to review here arose in the context of a new approach to topology, namely the Basic Picture (see [12]).

A *cover relation* over  $S$ , written  $\triangleleft$ , is a relation (between elements and subsets of  $S$ ) satisfying the following rules (written in natural-deduction style):

$$\frac{a \in U}{a \triangleleft U} \text{ reflexivity}, \quad \frac{\begin{array}{c} [b \in U] \\ \vdots \\ a \triangleleft U \quad b \triangleleft V \\ a \triangleleft V \end{array}}{a \triangleleft V} \text{ transitivity.} \quad (2)$$

We will use the word “cover” also for the structure  $(S, \triangleleft)$ . Often,  $U \triangleleft V$  will be used instead of  $(\forall b \in U)(b \triangleleft V)$ . Typically,  $S$  is a set of basic open subsets of a topological space and  $a \triangleleft U$  stands for “the basic open subset  $a$  is contained in the union of those belonging to  $U$ ”. In a natural way, one defines an operator on subsets, written  $\mathcal{A}$ , by  $\mathcal{A}U = \{a \in S : a \triangleleft U\}$ ; in other words,  $a \in \mathcal{A}U$  if and only if  $a \triangleleft U$ . It is easy to check that  $\mathcal{A}$  is a saturation (or closure) operator, which is to say that it satisfies the equivalence

$$U \subseteq \mathcal{A}V \leftrightarrow \mathcal{A}U \subseteq \mathcal{A}V \quad (3)$$

or, equivalently, all the following rules

$$\frac{}{U \subseteq \mathcal{A}U} \quad \frac{U \subseteq V}{\mathcal{A}U \subseteq \mathcal{A}V} \quad \frac{U \subseteq \mathcal{A}V}{\mathcal{A}U \subseteq \mathcal{A}V} \quad (4)$$

for any  $U$  and  $V$ . Vice versa, provided that  $\mathcal{A}$  is a saturation operator, the equation  $a \triangleleft U \equiv a \in \mathcal{A}U$  defines a cover relation; the correspondence between cover relations and closure operators is bijective. If  $U = \mathcal{A}U$ , then  $U$  is called *saturated* or *formal open*. The collection of all formal open subsets is written  $\text{Sat}(\mathcal{A})$  and is endowed with the equivalence relation  $=_{\mathcal{A}}$  where  $U =_{\mathcal{A}} V \equiv \mathcal{A}U = \mathcal{A}V$  (we use  $a =_{\mathcal{A}} U$  instead of  $\{a\} =_{\mathcal{A}} U$ ). As a consequence of its definition, a saturation operator satisfies a lot of interesting equations; the following one can be useful:

$$\mathcal{A} \bigcup_{i \in I} \mathcal{A}V_i = \mathcal{A} \bigcup_{i \in I} V_i \quad \text{that is} \quad \bigcup_{i \in I} \mathcal{A}V_i =_{\mathcal{A}} \bigcup_{i \in I} V_i. \quad (5)$$

It is well known that the collection of all fixed points of a closure operator is a complete lattice with respect to the following operations:

$$\bigvee_{i \in I} \mathcal{A}W_i = \mathcal{A} \bigcup_{i \in I} W_i \quad \bigwedge_{i \in I} \mathcal{A}W_i = \bigcap_{i \in I} \mathcal{A}W_i; \quad (6)$$

moreover,  $\mathcal{A}U \leq \mathcal{A}V$  if and only if  $U \triangleleft V$ .

From a topological point of view, one wants the intersection of two open subsets to be open too. This holds if “convergence”, which is linked to distributivity of  $\text{Sat}(\mathcal{A})$ , is fulfilled, that is if the following rule holds:

$$\frac{a \triangleleft U \quad a \triangleleft V}{a \triangleleft U \downarrow V} \downarrow \text{-Right} \quad (7)$$

where  $U \downarrow V = \bigcup_{u \in U, v \in V} (u \downarrow v)$  and  $u \downarrow v = \{b \in S : b \triangleleft \{u\} \text{ \& } b \triangleleft \{v\}\}$ . A cover relation satisfying  $\downarrow$ -Right is called *convergent*.

A *positivity relation* over  $S$ , written  $\ltimes$ , is a relation (between elements and subsets of  $S$ ) satisfying the following rules:

$$\frac{a \ltimes U}{a \ltimes U} \text{ co-reflexivity} \quad \frac{\begin{array}{c} [b \ltimes U] \\ \vdots \\ a \ltimes U \quad b \in V \\ a \ltimes V \end{array}}{a \ltimes V} \text{ co-transitivity.} \quad (8)$$

This notion has been introduced in [12] in order to solve a long-standing problem: to characterize closed subsets in a point-free way. The meaning of  $a \ltimes U$  is “there is a point in  $a$  such that all its basic neighbourhoods are in  $U$ ”. It is quite natural to introduce another operator on subsets, say  $\mathcal{J}$ , and define  $a \in \mathcal{J}U$  as  $a \ltimes U$ ; this is a reduction (or interior) operator, that is it satisfies:

$$\mathcal{J}U \subseteq V \leftrightarrow \mathcal{J}U \subseteq \mathcal{J}V. \quad (9)$$

Let us write  $\text{Red}(\mathcal{J})$  for the collection of all fixed points of  $\mathcal{J}$ ; the elements of this collection are called *formal closed* subsets.  $\text{Red}(\mathcal{J})$  is a complete lattice with respect to the operations

$$\bigvee_{i \in I} \mathcal{J}W_i \equiv \bigcup_{i \in I} \mathcal{J}W_i \quad \text{and} \quad \bigwedge_{i \in I} \mathcal{J}W_i \equiv \mathcal{J} \bigcap_{i \in I} W_i. \quad (10)$$

Conversely, provided that  $\mathcal{J}$  is a reduction operator, the relation  $a \ltimes U$  defined as  $a \in \mathcal{J}(U)$  is a binary positivity predicate.

We say that a cover and a positivity relation over the same set are *compatible* if the following rule is satisfied:

$$\frac{a \triangleleft U \quad a \ltimes V}{(\exists b \in U)(b \ltimes V)} \text{ compatibility.} \quad (11)$$

It is customary to write  $U \ltimes V$  instead of  $(\exists b \in U)(b \ltimes V)$ .

A *basic topology* is a triple  $(S, \triangleleft, \ltimes)$  where:

- $\triangleleft$  is a cover relation on  $S$ ;
- $\ltimes$  is a binary positivity predicate over  $S$ ;
- $\triangleleft$  and  $\ltimes$  are compatible.

Equivalently, a basic topology can be defined as a triple  $(S, \mathcal{A}, \mathcal{J})$ , where  $S$  is a set and the following hold for any  $U, V \subseteq S$

- ( $\mathcal{A}$  is a saturation operator on  $\mathcal{P}S$ )  $U \subseteq \mathcal{A}V \leftrightarrow \mathcal{A}U \subseteq \mathcal{A}V$ ;
- ( $\mathcal{J}$  is a reduction operator on  $\mathcal{P}S$ )  $\mathcal{J}U \subseteq V \leftrightarrow \mathcal{J}U \subseteq \mathcal{J}V$ ;
- ( $\mathcal{A}$  and  $\mathcal{J}$  are compatible)  $\mathcal{A}U \mathcal{J}V \leftrightarrow U \mathcal{J}V$ .

If classical logic is used, then compatibility will force  $\mathcal{A}U$  to be contained in  $\neg \mathcal{J} - U$  for all  $U$ ; analogously for  $\mathcal{J}$  and  $\neg \mathcal{A} -$ . Thus, classically speaking, a basic topology is just a set equipped with two operators on subsets which are either both saturation or both reduction operators and such that one is finer than the other.

**Definition 1.** A *formal topology* (or *convergent basic topology*) is a basic topology whose cover relation is convergent (that is, it satisfies (7)).

**Definition 2.** Let  $\mathcal{J} = (S, \triangleleft, \ltimes)$  be a formal topology. A *formal point* is a subset  $F \subseteq S$  which is:

- formal closed,
- inhabited,
- convergent, that is  $(a \in F \ \& \ b \in F) \rightarrow a \downarrow b \mathcal{J} F$  for any  $a, b \in S$ .

This predicative definition captures the impredicative notion of a completely prime filter over  $\text{Sat}(\mathcal{A})$  (see [12]). The collection of all formal points, written  $Pt(\mathcal{J})$ , is just the infinitary notion one wants to handle by means of the constructive definition of formal topology. Moreover,  $Pt(\mathcal{J})$  can be endowed with a natural topology whose basis is  $S$  itself: in fact,  $F$  is a point “in”  $a$  if  $a \in F$ . This influences the definition of morphism between two formal topologies. Let  $s$  be a relation between two sets  $S$  and  $T$ ; we define four operators between  $\mathcal{P}S$  and  $\mathcal{P}T$  in the following way:

- $sU = \{b \in T : (\exists a \in S)(asb \ \& \ a \in U)\}$ ;
- $s^*V = \{a \in S : (\forall b \in T)(asb \rightarrow b \in V)\}$ ;
- $s^-$  and  $s^{-*}$ : the same definitions with respect to the relation  $s^-$ , the inverse relation of  $s$ ;

(provided that  $a \in S$  and  $b \in T$ , we write  $sa$  and  $s^-b$  instead of  $s\{a\}$  and  $s^-\{b\}$ , respectively). In other words:

$$\begin{aligned} b \in sU &\equiv s^-b \mathcal{J} U & a \in s^-V &\equiv sa \mathcal{J} V \\ a \in s^*V &\equiv sa \subseteq V & b \in s^{-*}U &\equiv s^-b \subseteq U. \end{aligned} \quad (12)$$

**Definition 3.** Let  $\mathcal{J} = (S, \triangleleft, \ltimes)$  and  $\mathcal{J}' = (T, \triangleleft', \ltimes')$  be two basic topologies. A relation  $s$  between  $S$  and  $T$  is a *continuous relation* if it satisfies

$$\frac{b \triangleleft' V}{s^-b \triangleleft s^-V} \quad \text{and} \quad \frac{s^-b \ltimes s^*V}{b \ltimes' V} \quad (13)$$

for any  $b \in T$  and  $V \subseteq T$ . Two continuous relations  $s_1$  and  $s_2$  are declared to be equal if  $s_1^-b =_{\mathcal{A}} s_2^-b$ , for any  $b \in T$ .

Let  $\mathcal{J} = (S, \triangleleft, \ltimes)$  and  $\mathcal{J}' = (T, \triangleleft', \ltimes')$  be two formal topologies. A relation  $s$  between  $S$  and  $T$  is a *continuous map* if:

- $s$  is a *continuous relation*;
- $s$  is *convergent*, that is  $s^-a \downarrow s^-b =_{\mathcal{A}} s^-(a \downarrow b)$ , for any  $a, b \in T$ ;
- $s$  is *total*, that is  $s^-T =_{\mathcal{A}} S$ .

Basic topologies (formal topologies) and continuous relations (maps) form a category (see [12]). It can be shown (see [12]) that  $s$  is a continuous relation if and only if  $s^{-*}$  and  $s$  are a formal open function and a formal closed function, respectively. Two morphisms, say  $s_1$  and  $s_2$ , are equal if and only if the functions  $s_1^{-*}$  and  $s_2^{-*}$  are equal on formal open subsets and the functions  $s_1$  and  $s_2$  are equal on formal closed subsets. Moreover,  $s : S \rightarrow T$  is an isomorphism if there exists another continuous relation, say  $s'$ , from  $T$  to  $S$  such that the conditions

$$(s's)^-a =_{\mathcal{A}} a \quad (ss')^-b =_{\mathcal{A}'} b \quad (14)$$

hold for any  $a \in S$  and  $b \in T$  or, equivalently, all the following hold:

- $s's$  is the identity map on  $\text{Red}(\mathcal{J})$ ;

- $(s's)^{-*}$  is the identity map on  $\text{Sat}(\mathcal{A})$ ;
- $ss'$  is the identity map on  $\text{Red}(\mathcal{J}')$ ;
- $(ss')^{-*}$  is the identity map on  $\text{Sat}(\mathcal{A}')$ .

Note that each continuous map is just a continuous function (in the usual sense) between the topological spaces  $Pt(\mathcal{J})$  and  $Pt(\mathcal{J}')$ . Finally, if one is interested only in the cover relation, then one simply has to remove the condition on  $\ltimes$  and  $\ltimes'$  because every cover  $(S, \triangleleft)$  can be seen as a basic topology in which  $a \ltimes V$  is always false.

The definition of formal topology given above differs from the original one (see [11]) in two respects, at least. One is the introduction of the binary positivity  $\ltimes$  that replaces the (unary) positivity, written  $\text{Pos}$ . The other one is the absence of an operation on  $S$  that is replaced by  $\downarrow$ . In the present paper, we just need an intermediate notion.

**Definition 4.** A formal topology with operation is a structure  $(S, \triangleleft, \ltimes, \cdot)$ , where the triple  $(S, \triangleleft, \ltimes)$  is a basic topology and  $\cdot$  is a binary operation on  $S$  such that the following rules are fulfilled for any  $a, b \in S$  and  $U, V \subseteq S$ :

$$\frac{a \triangleleft U \quad a \triangleleft V}{a \triangleleft U \cdot V} (\cdot - \text{Right}) \quad \frac{a \triangleleft U}{a \cdot b \triangleleft U} \quad \frac{a \triangleleft U}{b \cdot a \triangleleft U} (\cdot - \text{Left}) \quad (15)$$

where  $U \cdot V = \{a \cdot b : a \in U, b \in V\}$ .

It is easily seen that  $a \cdot b =_{\mathcal{A}} a \downarrow b$  and  $U \cdot V =_{\mathcal{A}} U \downarrow V$ . Thus the definition of morphism (that is continuous map) between formal topologies with operation and that of formal point of a formal topology with operation are obtained by literally replacing  $\downarrow$  by  $\cdot$  in Definitions 2 and 3.

**Theorem 5.** Every formal topology is isomorphic to a formal topology with operation.

**Proof.** See [3].  $\square$

We use the name “cover with operation” for a structure  $(S, \triangleleft, \cdot)$ , where  $\triangleleft$  is a cover and  $\cdot$  is a binary operation on  $S$  satisfying  $\cdot - \text{Right}$  and  $\cdot - \text{Left}$  of Definition 4.

It could seem natural to ask for some additional properties about  $\cdot$  such as associativity or commutativity, but that is not really needed because of the following easy proposition.

**Proposition 6.** If  $(S, \triangleleft, \cdot)$  is a cover with operation, then  $(S, \cdot, =_{\mathcal{A}})$  is a semilattice (that is, an idempotent, commutative semigroup).

Moreover,  $(\text{Sat}(\mathcal{A}), \cdot, S)$  is a bounded semilattice (that is, an idempotent, commutative monoid).

An important class of basic topologies is that of generated ones (see [5] for the generation of formal topologies). Let  $\{I(a)\}_{a \in S}$  be a family of sets and  $C(a, i) \subseteq S$  for any  $a \in S$  and  $i \in I(a)$ : this is called an axiom-set over  $S$ . A cover relation can be generated, by induction, by means of the following rules:

$$\frac{a \triangleleft U \quad i \in I(a) \quad C(a, i) \triangleleft U}{a \triangleleft U} \quad \frac{[b \in U] \quad [i \in I(b), C(b, i) \subseteq P]}{a \triangleleft U \quad b \in P \quad b \in P} \text{induction.} \quad (16)$$

Informally, we say that  $\triangleleft$  is the smallest cover relation satisfying the axioms  $a \triangleleft C(a, i)$ . On the other hand, a positivity relation can be generated, by co-induction, in the following way (see [9]):

$$\frac{a \ltimes V \quad i \in I(a) \quad a \ltimes V}{a \ltimes V} \quad \frac{[b \in P] \quad [b \in P, i \in I(b)]}{a \in P \quad b \in V \quad C(b, i) \not\ltimes P} \text{co-induction.} \quad (17)$$

It can be proved that the two relations  $\triangleleft$  and  $\ltimes$  generated by the above rules are indeed a cover and a positivity relation, respectively, and satisfy the compatibility rule. This positivity relation turns out to be the maximal one which is compatible with the corresponding inductively generated cover; in other words,  $\mathcal{J} = -\mathcal{A}-$  provided that classical logic is assumed. However, one can generate a cover and a positivity relation quite independently one from another as it is shown by the following proposition we give without proof.

**Proposition 7.** Let  $(I, C)$  and  $(J, D)$  be two axiom-sets on a set  $S$ . Let  $\triangleleft_{I,C}$  be the cover relation generated by means of the axiom set  $(I, C)$ . Let  $(I + J, C + D)$  be the axiom set obtained as the disjoint union of  $(I, C)$  and  $(J, D)$  and let  $\ltimes_{I+J, C+D}$  be the binary positivity predicate generated by means of it. Then  $\triangleleft_{I,C}$  and  $\ltimes_{I+J, C+D}$  are compatible.

#### 4. Finitary cover relations

**Definition 8.** A cover relation  $\triangleleft$  on a set  $S$  is *finitary* if

$$a \triangleleft U \rightarrow (\exists K \subseteq_{\omega} S)(a \triangleleft K \subseteq_{\omega} U) \quad (18)$$

for any  $a \in S$  and  $U \subseteq S$ .

There are several simple examples of finitary cover relations.

**Theorem 9.** Let  $\triangleleft$  be the cover relation generated by an axiom-set  $(I, C)$  such that each  $C(a, i)$  is finite; then  $\triangleleft$  is a finitary cover.

**Proof.** The proof is an easy induction with  $P = \{b \in S : (\exists K \subseteq_{\omega} U)(b \triangleleft K)\}$ . If  $b \in U$ , then put  $K = \{b\}$ ; of course  $K \subseteq_{\omega} U$  and  $b \triangleleft K$ ; thus  $b \in P$ . Instead, if  $i \in I(b)$  and  $C(b, i) \subseteq P$ , then argue as follows. Since  $C(b, i)$  is finite, we can decide whether it is empty or inhabited. In the first case put  $K = \emptyset$ . Otherwise, let  $C(b, i) = \{b_1, \dots, b_n\}$ ,  $n \geq 1$ ; so  $C(b, i) \subseteq P$  means that  $b_i \in P$  for each  $i$ . Thus, for each  $i$ , we are able to compute  $L_i$  which satisfies  $b_i \triangleleft L_i \subseteq_{\omega} U$ . Then it is enough to put  $K = L_1 \cup \dots \cup L_n$  which is surely a finite subset of  $U$ . Moreover, since  $b_i \triangleleft L_i$  for any  $i$ , then  $b_i \triangleleft K$  for any  $i$ , that is  $C(b, i) \triangleleft K$ , from which  $b \triangleleft K$  follows.  $\square$

**Corollary 10.** If  $\triangleleft$  is generated by  $(I, C)$ , then the cover  $\triangleleft_{\text{fin}}$  generated by means of those  $C(a, i)$  which are finite, is finitary and is contained in  $\triangleleft$  (that is  $a \triangleleft_{\text{fin}} U \rightarrow a \triangleleft U$ ).

**Proposition 11.** Let  $\triangleleft$  be a cover relation on  $S$  generated by  $(I, C)$  and suppose that it is finitary. Then there exists an axiom-set  $(I, D)$  which generates  $\triangleleft$  and such that each  $D(a, i)$  is finite.

**Proof.** For each  $a \in S$  and  $i \in I(a)$ , let  $D(a, i)$  be a finite subset such that  $a \triangleleft D(a, i) \subseteq_{\omega} C(a, i)$  (note that  $a \triangleleft C(a, i)$  and use the fact that  $\triangleleft$  is finitary). Let  $\triangleleft'$  be the cover relation generated by  $(I, D)$ . We want to show that  $a \triangleleft U \Leftrightarrow a \triangleleft' U$ .

$\Rightarrow$ ) Suppose that  $a \triangleleft U$ ; then either  $a \in U$  and we are done, or there is an index  $i$  in  $I(a)$  such that  $C(a, i) \triangleleft U$ . Hence  $C(a, i) \triangleleft' U$  by induction. But  $D(a, i) \subseteq C(a, i)$  hence  $D(a, i) \triangleleft' U$  and we can conclude.

$\Leftarrow$ ) Vice versa, suppose that  $a \triangleleft' U$ ; if  $a \in U$  we are done. Otherwise, there exists  $i \in I(a)$  such that  $D(a, i) \triangleleft' U$ . By induction, we have  $D(a, i) \triangleleft U$  too. Consider the subset  $C(a, i)$  corresponding to the same index  $i$ ; surely  $a \triangleleft C(a, i)$ , so  $a \triangleleft D(a, i)$  by definition of  $D$ ;  $a \triangleleft U$  follows by transitivity.  $\square$

The following important theorem states that, in order to study finitary cover relations, it is enough to restrict one's attention to the generated ones.

**Theorem 12.** Any finitary cover relation can be generated by an axiom-set.

**Proof.** See [5]. See also Proposition 19, item 3.  $\square$

The following fact follows at once from Proposition 11, Theorems 9 and 12.

**Corollary 13.** Finitary cover relations are exactly those cover relations that can be generated by an axiom-set  $(I, C)$  such that each  $C(a, i)$  is finite.

**Definition 14.** Let  $\triangleleft$  be an arbitrary cover relation. The relation

$$a \triangleleft_{\omega} U \equiv (\exists K \in \mathcal{P}_{\omega} U)(a \triangleleft K) \quad (19)$$

is called the *finitarization* of  $\triangleleft$ .

It is easily seen that  $\triangleleft_{\omega}$  is indeed a finitary cover. Provided that  $\triangleleft$  and  $\triangleleft'$  are two cover relations over the same set  $S$ , we write  $\triangleleft' \subseteq \triangleleft$  to mean that  $a \triangleleft' U \rightarrow a \triangleleft U$ , for any  $a \in S$  and  $U \subseteq S$ .

**Proposition 15.** For any two cover relations  $\triangleleft$  and  $\triangleleft'$  over the same set  $S$ , the following are equivalent:

- a.  $\triangleleft'$  is equal to  $\triangleleft_{\omega}$ , the finitarization of  $\triangleleft$ ;
- b.  $\triangleleft'$  is a finitary cover,  $\triangleleft' \subseteq \triangleleft$  and, provided that  $K$  is finite, if  $a \triangleleft K$ , then  $a \triangleleft' K$ ;
- c.  $\triangleleft'$  is a finitary cover,  $\triangleleft' \subseteq \triangleleft$  and for any finitary cover  $\triangleleft''$ , if  $\triangleleft'' \subseteq \triangleleft$ , then  $\triangleleft'' \subseteq \triangleleft'$ ; in other words,  $\triangleleft'$  is the greatest finitary cover contained in  $\triangleleft$ .

**Proof.** a.  $\Rightarrow$  b.  $\triangleleft'$  is finitary since it is equal to  $\triangleleft_{\omega}$ ; as  $\triangleleft_{\omega} \subseteq \triangleleft$ , then also  $\triangleleft' \subseteq \triangleleft$ . If  $a \triangleleft K$  with  $K$  finite, then  $a \triangleleft_{\omega} K$  and hence  $a \triangleleft' K$ .

b.  $\Rightarrow$  c. It is enough to show that if  $\triangleleft''$  is finitary and  $\triangleleft'' \subseteq \triangleleft$ , then  $\triangleleft'' \subseteq \triangleleft'$ . Assume that  $a \triangleleft'' U$ ; since  $\triangleleft''$  is finitary, then  $a \triangleleft'' K$  for some  $K \subseteq_{\omega} U$ , hence  $a \triangleleft K$  because  $\triangleleft'' \subseteq \triangleleft$ ; so  $a \triangleleft' K$  by b. and therefore  $a \triangleleft' U$ .

c.  $\Rightarrow$  a. Let  $a \triangleleft' U$ ; since  $\triangleleft'$  is finitary, then  $a \triangleleft' K$  for some  $K \subseteq_{\omega} U$  and hence  $a \triangleleft K$  because  $\triangleleft' \subseteq \triangleleft$ ; so  $a \triangleleft_{\omega} U$  and  $\triangleleft' \subseteq \triangleleft_{\omega}$ . Conversely,  $\triangleleft_{\omega} \subseteq \triangleleft$  and  $\triangleleft_{\omega}$  is finitary by definition; so  $\triangleleft_{\omega} \subseteq \triangleleft'$  by c.  $\square$

Thus,  $\triangleleft_{\omega}$  is the greatest finitary cover relation among those which are contained in  $\triangleleft$ .

**Corollary 16.** A cover relation is finitary if and only if it coincides with its finitarization.

#### 4.1. Finitary bases

Now we start a different approach to finitary covers (we follow essentially the same idea as in [10]).

**Definition 17.** Let  $\triangleleft$  be a cover. The relation  $\prec_{\triangleleft}$  between elements and finite subsets defined by  $a \prec_{\triangleleft} K \equiv a \triangleleft K$  is called the *finitary trace* of  $\triangleleft$ .

First of all, we want to find an axiomatization of the concept of finitary trace.

**Definition 18.** A relation  $\prec$  between element of  $S$  and finite subset of  $S$  is called a *finitary base* if it satisfies:

$$\frac{a \in K}{a \prec K} \text{ reflexivity} \quad \text{and} \quad \frac{a \prec K \quad K \prec L}{a \prec L} \text{ transitivity} \quad (20)$$

where  $K \prec L \equiv (\forall b \in K)(b \prec L)$ .

It is quite trivial to check that the finitary trace  $\prec_{\triangleleft}$  associated to a cover  $\triangleleft$  is a finitary base. Vice versa, if  $\prec$  is a finitary base on a set  $S$ , then a cover, say  $\triangleleft_{\prec}$ , can be generated by means of the following rules (see [5]):

$$\frac{a \in U}{a \triangleleft_{\prec} U} \text{ (reflexivity)} \quad \frac{a \prec K \quad K \triangleleft_{\prec} U}{a \triangleleft_{\prec} U} \text{ (transitivity on } \prec \text{ --axioms).} \quad (21)$$

Note that if one writes  $R(a, K)$  instead of  $a \prec K$  the latter rule becomes the “transitivity on  $\prec$ -axioms” of [5]; transitivity on  $\prec$ -axioms is allowed because  $K$  ranges over a set-indexed family. Since each  $C(a, i)$  is finite,  $\triangleleft_{\prec}$  is finitary. Note that  $\triangleleft_{\prec}$  is the smallest cover relation satisfying  $a \triangleleft_{\prec} K$  whenever  $a \prec K$ . Formally, we can say that  $\triangleleft_{\prec}$  is generated by the axiom-set  $(I, C)$  where  $I(a) = \{K : a \prec K\}$  and  $C(a, K) = K$ .

**Proposition 19.** For each finitary base  $\prec$  and each cover  $\triangleleft$ , the following hold:

1.  $\prec_{\triangleleft_{\prec}} = \prec$ ;
2.  $\triangleleft_{\prec_{\triangleleft}} = \triangleleft_{\omega}$ ;
3.  $\triangleleft$  is finitary if and only if  $\triangleleft_{\prec_{\triangleleft}} = \triangleleft$ .

**Proof.** Let us prove item 1. From  $a \prec K$  one can prove  $a \triangleleft_{\prec} K$  by transitivity on  $\prec$ -axioms ( $K \triangleleft_{\prec} K$  follows by reflexivity) and then  $a \prec_{\triangleleft_{\prec}} K$  because  $K$  is finite. Vice versa, if  $a \prec_{\triangleleft_{\prec}} K$ , then  $a \triangleleft_{\prec} K$  and  $K \subseteq_{\omega} S$  by definition;  $a \prec K$  can be proved by induction on the proof of  $a \triangleleft_{\prec} K$ , as follows: if  $a \in K$ , then one uses reflexivity of  $\prec$ ; if  $a \prec L$  and  $L \triangleleft_{\prec} K$  then argue as follows: provided that  $L = \{b_1, \dots, b_n\}$ ,  $L \triangleleft_{\prec} K$  means that  $b_i \triangleleft_{\prec} K$  for any  $i \leq n$ ; by inductive hypothesis,  $b_i \prec K$  for each  $i \leq n$ , that is  $L \prec K$ ; this fact together with  $a \prec L$  gives  $a \prec K$  by transitivity of the finitary base  $\prec$ .

Now we can prove item 2.

$$\begin{aligned} a \triangleleft_{\prec_{\triangleleft}} V &\equiv (\exists K \subseteq_{\omega} V)(a \triangleleft_{\prec_{\triangleleft}} K) \equiv && \text{because } \triangleleft_{\prec_{\triangleleft}} \text{ is finitary;} \\ &\equiv (\exists K \subseteq_{\omega} V)(a \prec_{\triangleleft_{\triangleleft}} K) \equiv && \text{by definition of } \prec_{\triangleleft}; \\ &\equiv (\exists K \subseteq_{\omega} V)(a \prec_{\triangleleft} K) \equiv && \text{from item 1;} \\ &\equiv (\exists K \subseteq_{\omega} V)(a \triangleleft K) \equiv && \text{by definition of } \prec_{\triangleleft}; \\ &\equiv a \triangleleft_{\omega} V && \text{by definition of } \triangleleft_{\omega}. \end{aligned}$$

Item 3. follows from item 2. and Corollary 16.  $\square$

Note that Theorem 12 follows from item 3., that is  $\triangleleft$  is finitary if and only if it coincides with the cover generated by its finitary trace. Note also that, if  $\triangleleft$  is generated by  $(I, C)$ , then  $\triangleleft_{\omega}$  does not need to coincide with  $\triangleleft_{fin}$ , that is the cover generated by those  $C(a, i)$  which are finite. Indeed, let  $S = \{a, b\} \cup \mathbb{N}$ ,  $a \triangleleft \mathbb{N}$  and  $n \triangleleft \{b\}$  for all  $n \in \mathbb{N}$ . Hence  $a \triangleleft \{b\}$  follows by transitivity, but one cannot prove it without using the axiom  $a \triangleleft \mathbb{N}$ . Thus  $a \triangleleft_{fin} \{b\}$  does not hold, even if  $a \triangleleft_{\omega} \{b\}$  is true.

Finally, if  $(S, \triangleleft, \cdot)$  is a cover with operation, then its finitary trace  $\prec_{\triangleleft}$  satisfies the additional conditions

$$\frac{a \prec K \quad a \prec L}{a \prec K \cdot L} \quad \frac{a \prec K}{a \cdot b \prec K} \quad \frac{a \prec K}{b \cdot a \prec K} \quad (22)$$

which we can call  $\cdot$  — Right and  $\cdot$  — Left. Vice versa, it can be proved that the cover generated by a finitary base satisfying (22) actually is a finitary cover with operation. A finitary base which satisfies (22) is called a *finitary base with operation*. Thus finitary covers with operation are exactly those cover relations which are generated by means of finitary bases with operation.



## 5. Finitary topologies

In the previous sections we have analyzed the cover relation  $\triangleleft$ , now we want to analyze the positivity relation too. In other words, we are going to study the finitization of a topology and not only of a cover.

**Proposition 20.** *If  $(S, \triangleleft, \times)$  is a basic topology and  $\triangleleft'$  is a subcover of  $\triangleleft$  (that is  $\triangleleft' \subseteq \triangleleft$ ), then  $(S, \triangleleft', \times)$  is still a basic topology.*

**Proof.** We only need to verify compatibility between  $\triangleleft'$  and  $\times$ : from  $a \triangleleft' U$  one gets  $a \triangleleft U$  and then the thesis by compatibility between  $\triangleleft$  and  $\times$ .  $\square$

**Corollary 21.** *For any basic topology  $\mathcal{S} = (S, \triangleleft, \times)$ , the structure  $\mathcal{S}_\omega = (S, \triangleleft_\omega, \times)$  is still a basic topology.*

Long time ago, both T. Coquand and S. Valentini noted that if  $(S, \triangleleft, \cdot, 1, Pos)$  is a formal topology according to the original definition in [11], then the structure  $(S, \triangleleft_\omega, \cdot, 1, Pos)$  is a formal topology only if the predicate  $Pos$  is decidable (this fact can be seen as a corollary of [14], Proposition 12; see also [6], Proposition 3.6 and Corollary 3.7). Thus the previous corollary shows that the binary positivity predicate works better than the unary one, at least with respect to the process of finitization.

As we said above, provided that  $\triangleleft$  is a cover relation, we are able to construct (following the construction in [5]) an axiom-set by means of the finitary trace  $\prec_{\triangleleft}$ . This axiom-set allows us to generate the finitization  $\triangleleft_\omega$  of the cover, but also a binary positivity relation, say  $\times_\omega$ . In general, provided that  $\prec$  is a finitary base, we can generate (co-inductively) a binary positivity predicate, say  $\times_\prec$ , by means of the axiom-set corresponding to  $\prec$ . Explicitly, the rules generating  $\times_\prec$  becomes:

$$\frac{a \times_\prec V}{a \in V} \quad \frac{a \prec K \quad a \times_\prec V}{K \times_\prec V} \quad \frac{\begin{array}{c} [b \in P] \quad [b \in P, b \prec K] \\ | \quad | \\ a \in P \quad b \in V \quad K \not\prec P \\ a \times_\prec V \end{array}}{a \times_\prec V}. \quad (23)$$

Now, we can define  $\times_\omega$  as  $\times_{\prec_{\triangleleft}}$  and consider the structure  $(S, \triangleleft_\omega, \times_\omega)$  which is a basic topology too. So, it seems natural to give the following definition.

**Definition 22.** Let  $\mathcal{S} = (S, \triangleleft, \times)$  be a basic topology; then  $\mathcal{S}_\omega = (S, \triangleleft_\omega, \times_\omega)$  (that is the basic topology generated by (the axiom-set corresponding to)  $\prec_{\triangleleft}$ ) is called the *finitization* of  $\mathcal{S}$ . A basic topology is *finitary* if it coincides with its finitization.

Note that the finitization of a basic topology is completely described by its cover, since both  $\triangleleft_\omega$  and  $\times_\omega$  are determined by  $\prec_{\triangleleft}$ . If  $\prec$  is the finitary trace of  $\triangleleft$ , then  $a \triangleleft_\omega U$  is equivalent to  $(\exists K \in \mathcal{P}_\omega S)(a \prec K \ \& \ K \subseteq U)$ , as we know by the discussion in the previous section. The question poses itself whether  $\times_\omega$  can be characterized in a similar explicit way. Actually, the answer is affirmative only if classical logic is used; in that case one has:  $a \times_\omega V$  if and only if  $(\forall K \in \mathcal{P}_\omega S)(a \prec K \rightarrow K \not\prec V)$ . There exists an impredicative way to define a finitary base corresponding to a given binary positivity relation  $\times$ : it is enough to put

$$a \prec_\times K \equiv (\forall U \subseteq S)(a \times U \rightarrow K \times U) \quad (24)$$

for any  $a \in S$  and  $K \in \mathcal{P}_\omega S$ . Actually, that base is the trace of the maximal cover compatible with  $\times$ . Thus, working in a classical and impredicative foundation allows to define  $\times_\omega$  in terms of  $\times$  in the following way:

$$a \times_\omega V \equiv (\forall K \in \mathcal{P}_\omega S) \left( (\forall U \subseteq S)(a \times U \rightarrow K \times U) \rightarrow K \not\prec V \right). \quad (25)$$

However, the relation  $\times_\omega$  co-generated by means of  $\prec_{\triangleleft}$  is the best way to approximate the classical situation in our framework.

Now we want to describe morphisms between finitary basic topologies and introduce a notion of morphism between sets equipped with finitary bases in such a way that the two corresponding categories would become equivalent.

**Definition 23.** Let  $\mathcal{S} = (S, \triangleleft, \times)$  and  $\mathcal{T} = (T, \triangleleft', \times')$  be two finitary basic topologies; a relation  $s$  between  $S$  and  $T$ ;  $s$  is a *coherent continuous relation* if

- (1)  $s$  is a continuous relation (see Definition 3);
- (2)  $s^-b \in (\mathcal{P}_\omega S / =_{\mathcal{A}})$ , for any  $b \in T$ ; that is, there exists  $K \subseteq_\omega S$  such that  $s^-b =_{\mathcal{A}} K$ .

Two such relations, say  $s_1$  and  $s_2$ , are considered to give rise to the same morphism if they are equal as continuous relations.

**Definition 24.** Let  $(S, \prec)$  and  $(T, \prec')$  be two sets equipped with finitary bases and let  $s$  be a relation between  $S$  and  $T$ . Then  $s$  is a *morphism* between  $(S, \prec)$  and  $(T, \prec')$  if the following conditions hold:

$$s^-b \in (\mathcal{P}_\omega S / =_{\mathcal{A}}) \quad \frac{b \prec' K}{s^-b \prec s^-K} \quad (26)$$

for any  $b \in T$  and  $K \in \mathcal{P}_\omega T$ , where  $\mathcal{A}$  is the closure operator corresponding to the cover  $\triangleleft_\prec$ .

Two such relations, say  $s_1$  and  $s_2$ , are considered to give rise to the same morphism if  $s_1^-b \prec s_2^-b$  and  $s_2^-b \prec s_1^-b$  for any  $b \in T$ .



It is quite easy to prove that finitary basic topologies and coherent continuous relations form a category which is called **FBTop**. Similarly, finitary bases and their morphisms form a category called **FB**.

**Theorem 25.** *The categories **FBTop** and **FB** are equivalent.*

**Proof.** The proof is essentially trivial. For example, the bijection between objects is a corollary of 12, 16 and 19. Moreover, a coherent continuous relation between  $(S, \triangleleft, \bowtie)$  and  $(T, \triangleleft', \bowtie')$  is itself a morphism between the corresponding  $(S, \triangleleft)$  and  $(T, \triangleleft')$  because  $a \triangleleft_{\triangleleft} K$  is equivalent to  $a \triangleleft K$ , provided that  $K$  is finite. Vice versa, let  $s$  be a morphism between  $(S, \triangleleft)$  and  $(T, \triangleleft')$ . In order to check that  $s$  itself is a coherent continuous relation between  $(S, \triangleleft, \bowtie)$  and  $(T, \triangleleft', \bowtie')$ , the only perhaps non trivial step is to prove

$$\frac{s^-b \bowtie_{\triangleleft} s^*V}{b \bowtie_{\triangleleft'} V} \quad (27)$$

that can be done by *co-induction* on  $\bowtie_{\triangleleft'}$  as follows (see also [9]). Put  $b \in P \equiv s^-b \bowtie_{\triangleleft} s^*V$  and suppose that  $b \in P$ ; then there exists  $a \in S$  such that  $b \in sa$  (because  $a \in s^-b$ ) and  $sa \subseteq V$  (because  $a \in s^*V$ ); so  $b \in V$ . Summing up we have proved that  $b \in P$  implies  $b \in V$ , which is the first assumption of the co-induction rule. To conclude we have to prove that if  $b \in P$  and  $b \triangleleft' K$ , where  $K \in \mathcal{P}_\omega T$ , then  $K \bowtie P$ . Here is a sketch of the proof.

$$\frac{\frac{\frac{b \triangleleft' K}{s^-b \triangleleft s^-K} \quad \frac{b \in P}{s^-b \bowtie_{\triangleleft} s^*V}}{s^-K \bowtie_{\triangleleft} s^*V}}{(\exists c \in K)(s^-b \bowtie_{\triangleleft} s^*V)} \quad K \bowtie P \quad \square \quad (28)$$

The previous theorem allows us to simplify the notion of morphism in **FBTop**. In fact, a coherent continuous relation between  $\mathcal{S}$  and  $\mathcal{T}$  becomes simply a relation  $s$  between  $S$  and  $T$  satisfying

$$s^-b \in (\mathcal{P}_\omega S / =_{\mathcal{A}}) \quad \frac{b \triangleleft' K}{s^-b \triangleleft s^-K} \quad (29)$$

for any  $b \in T$  and  $K \in \mathcal{P}_\omega T$ .

**Corollary 21** does not hold in general for formal topologies, that is, one cannot prove  $\downarrow - \text{Right}$  (see display (7)) for  $\triangleleft_\omega$  from the assumption that  $\triangleleft$  satisfies  $\downarrow - \text{Right}$ . Indeed, let us consider the set  $S = \{a, b_1, b_2, c_1, c_2\} \cup \mathbb{N}$  and the cover generated by  $a \triangleleft \mathbb{N}$ ,  $2n \triangleleft \{b_1\}$ ,  $2n \triangleleft \{c_1\}$ ,  $2n+1 \triangleleft \{b_2\}$ ,  $2n+1 \triangleleft \{c_2\}$ , for any  $n \in \mathbb{N}$ . It is easy to check that  $\triangleleft$  satisfies  $\downarrow - \text{Right}$ , but  $a \triangleleft_\omega \{b_1, b_2\} \downarrow \{c_1, c_2\}$  is not true even if  $a \triangleleft_\omega \{b_1, b_2\}$  and  $a \triangleleft_\omega \{c_1, c_2\}$ ; that happens because  $\{b_1, b_2\} \downarrow \{c_1, c_2\} = \mathbb{N}$ .

In contrast, if the formal topology is with operation, then also  $\triangleleft_\omega$  satisfies  $\cdot - \text{Left}$  (trivially) and  $\cdot - \text{Right}$  because the  $\cdot$  of two finite subset is finite too; that explains why we considered formal topologies with operation.

**Definition 26.** Let  $\mathcal{S} = (S, \triangleleft, \bowtie, \cdot)$  be a formal topology with operation. Then the *finitarization* of  $\mathcal{S}$  is  $\mathcal{S}_\omega = (S, \triangleleft_\omega, \bowtie_\omega, \cdot)$ , where  $(S, \triangleleft_\omega, \bowtie_\omega)$  is the finitarization of  $(S, \triangleleft, \bowtie)$ . A formal topology with operation is *finitary* if it coincides with its finitarization.

Of course, in the case of two finitary formal topologies with operation, we have to add an obvious convergence condition to the definition of morphism.

**Definition 27.** A *coherent continuous map* between two finitary formal topologies with operation  $\mathcal{S} = (S, \triangleleft, \bowtie, \cdot)$  and  $\mathcal{T} = (T, \triangleleft', \bowtie', \cdot')$  is a relation  $s$  between  $S$  and  $T$  such that:

(1)  $s$  is a *coherent continuous relation* between  $\mathcal{S} = (S, \triangleleft, \bowtie, \cdot)$  and  $\mathcal{T} = (T, \triangleleft', \bowtie', \cdot')$ , that is

$$s^-b \in (\mathcal{P}_\omega S / =_{\mathcal{A}}) \quad \frac{b \triangleleft' K}{s^-b \triangleleft s^-K} \quad (30)$$

for any  $b \in T$  and  $K \in \mathcal{P}_\omega T$ ;

(2)  $s$  satisfies *convergence*, that is  $s^-a \cdot s^-b =_{\mathcal{A}} s^-(a \cdot' b)$ , for any  $a, b \in T$ ;

(3)  $s$  satisfies *totality*, that is  $s^-T =_{\mathcal{A}} S$ .

Analogously to the case of finitary basic topologies and finitary bases, one can easily prove that the category of finitary formal topologies with operation is equivalent to the category of finitary bases with operation (with the obvious notion of morphism).

## 6. Compactness and Stone's representation theorem

An element, say  $a$ , in a complete lattice  $(L, \leq)$  is called *compact* if for any (set-indexed) family  $\{b_i\}_{i \in I}$  of elements of  $L$  the following implication holds

$$a \leq \bigvee_{i \in I} b_i \rightarrow a \leq \bigvee_{i \in I_0} b_i \quad (31)$$

for some  $I_0 \subseteq_\omega I$ . In the particular case of  $\text{Sat}(\mathcal{A})$  (the lattice of formal open subsets of a cover  $(S, \triangleleft)$ ) the compactness of  $U$  becomes

$$U \triangleleft \bigcup_{i \in I} V_i \rightarrow U \triangleleft \bigcup_{i \in I_0} V_i \quad (32)$$

for some  $I_0 \subseteq_\omega I$ , where  $\{V_i\}_{i \in I}$  is an arbitrary set-indexed family of subsets of  $S$ .

**Lemma 28.** *The following are equivalent:*

1.  $U$  is a compact element in  $\text{Sat}(\mathcal{A})$ ;
2. for any  $V \subseteq S$ , there exists  $V_0 \subseteq_\omega V$  such that  $U \triangleleft V \rightarrow U \triangleleft V_0$ .

**Proof.** 1.  $\Rightarrow$  2. If  $U \triangleleft V$ , that is  $U \triangleleft \bigcup_{v \in V} \{v\}$ , then  $U \triangleleft \bigcup_{v \in V_0} \{v\}$ , for some  $V_0 \subseteq_\omega V$ , that is  $U \triangleleft V_0$ .

2.  $\Rightarrow$  1. If  $U \triangleleft \bigcup_{i \in I} V_i$ , then  $U \triangleleft V_0$  for some  $V_0 \subseteq_\omega \bigcup_{i \in I} V_i$ . Hence  $V_0 \subseteq \bigcup_{i \in I_0} V_i$  for some  $I_0 \subseteq_\omega I$ ;  $U \triangleleft \bigcup_{i \in I_0} V_i$  follows.  $\square$

Let us write  $K(\mathcal{A})$  for the collection of all compact elements in  $\text{Sat}(\mathcal{A})$ . We will use the expression “ $U$  is a compact subset of  $(S, \triangleleft)$ ” as a synonym for “ $U$  is a compact element of  $\text{Sat}(\mathcal{A})$ ”.

Note that a finite subset does not need to be compact; as an example, let us consider an infinitary-branching node in a tree equipped with the cover arising from the order. However the following proposition holds.

**Proposition 29.** *Let  $(S, \triangleleft)$  be a cover and let  $\mathcal{A}$  be the corresponding saturation operator. Then for any  $U, V \subseteq S$  the following hold:*

- a. if  $U =_{\mathcal{A}} V$ , then  $U$  is compact if and only if  $V$  is compact;
- b. if  $U$  is compact, then  $U =_{\mathcal{A}} U_0$ , for some  $U_0 \subseteq_\omega U$ ;
- c.  $\emptyset$  is compact;
- d. if  $U$  and  $V$  are compact, then  $U \cup V$  is compact.

**Proof.** a. By transitivity of  $\triangleleft$ , since  $U \triangleleft V$  and  $V \triangleleft U$ .

b.  $U \triangleleft U_0$ , for some  $U_0 \subseteq_\omega U$ , follows from  $U \triangleleft U$ ; vice versa,  $U_0 \triangleleft U$  by reflexivity. So  $U =_{\mathcal{A}} U_0$ .

c.  $\emptyset$  always is covered by  $\emptyset$  which is a finite subset of any  $V$ .

d. If  $U \cup V \triangleleft W$ , then  $U \triangleleft W$  and  $V \triangleleft W$ ; those imply  $U \triangleleft W_1$  and  $V \triangleleft W_2$  for some  $W_1, W_2 \subseteq_\omega W$ . Hence  $U \triangleleft W_1 \cup W_2$  and  $V \triangleleft W_1 \cup W_2$ , that is  $U \cup V \triangleleft W_1 \cup W_2$ . Moreover,  $W_1 \cup W_2$  is a finite subset of  $W$ .  $\square$

By Lemma 28, it is easy to see that a cover is finitary if and only if each element of  $S$ , seen as a singleton, is compact. From that fact and from items c. and d. in the previous proposition the following corollary follows.

**Corollary 30.** *A cover relation over a set  $S$  is finitary if and only if each finite subset of  $S$  is compact.*

In other words, provided that  $\triangleleft$  is a finitary cover on  $S$ , finite subsets and compact ones coincide with respect to  $=_{\mathcal{A}}$ . In this case  $K(\mathcal{A})$  can be identified with  $\mathcal{P}_\omega S / =_{\mathcal{A}}$ .

*Remark-* Item d. also says that the  $\vee$  of two compact elements of  $\text{Sat}(\mathcal{A})$  is compact too. On the contrary, the  $\wedge$  of two compact elements does not need to be compact. For example, let  $S = \mathbb{N} \cup \{a, b, c\}$  and let  $\triangleleft$  be the smallest cover relation on  $S$  which satisfies  $a \triangleleft \mathbb{N}$ ,  $a \triangleleft \{b\}$  and  $a \triangleleft \{c\}$ . Then  $\{b\}$  and  $\{c\}$  are compact but their  $\wedge$ , that is  $\{a\}$  is not (since it is covered by  $\mathbb{N}$  and by none of its finite subsets).

Again, we need to restrict our attention to cover relations with operation if we want  $K(\mathcal{A})$  to be closed under  $\wedge$ .

**Proposition 31.** *Let  $(S, \triangleleft, \cdot)$  be a finitary cover with operation; then the  $\wedge$  of two compact elements is compact too.*

**Proof.** Let  $U$  and  $V$  be compact; then  $U =_{\mathcal{A}} U_0$  and  $V =_{\mathcal{A}} V_0$ ,  $U_0$  and  $V_0$  finite. As a consequence we have

$$\mathcal{A}U \wedge \mathcal{A}V = \mathcal{A}U_0 \wedge \mathcal{A}V_0 = \mathcal{A}U_0 \cap \mathcal{A}V_0 = \mathcal{A}(U_0 \cdot V_0) =_{\mathcal{A}} U_0 \cdot V_0. \quad (33)$$

But  $U_0 \cdot V_0 = \{u \cdot v : u \in U_0, v \in V_0\}$  is finite and then it is compact, because  $\triangleleft$  is finitary (Corollary 30).  $\square$

**Remark-** If  $S$  is a convergent cover (without  $\cdot$ , but satisfying  $\downarrow - \text{Right}$ ), then  $S$  being finitary is not enough to prove that the  $\wedge$  of two compact subsets is compact too. Indeed, let  $Q$  and  $Q'$  be two copies of the set of rational numbers and let  $S$  be the set obtained by identifying the elements of  $Q$  and  $Q'$  below  $\sqrt{2}$  (with respect to the standard order); finally, let  $\triangleleft$  be the finitary cover over  $S$  induced by  $\leq$  (that is  $a \triangleleft U \equiv (\exists u \in U)(a \leq u)$ ). It is easy to verify that  $\triangleleft$  satisfies  $\downarrow - \text{Right}$ ; moreover, every finite subset is compact (because the cover is finitary). Consider the compact subsets  $\{2\}$  and  $\{2'\}$ , where  $2' \in Q'$  is the copy of 2;  $\mathcal{A}2 \cap \mathcal{A}2' = 2 \downarrow 2' = \{q \in S : q < \sqrt{2}\}$  which is not compact, because it is covered by itself, but it cannot be covered by a finite subset of its (because of the density of the rational numbers).

For the rest of the section, we will consider only finitary formal topologies with operation. We identify  $\text{Sat}(\mathcal{A})$  with  $\mathcal{P}S / \equiv_{\mathcal{A}}$ ; then  $\wedge$  and  $\vee$  between formal opens are identified with  $\cdot$  and  $\cup$  between subsets of  $S$ , while  $\leq$  in  $\text{Sat}(\mathcal{A})$  corresponds to  $\triangleleft$ . Similarly for  $K(\mathcal{A})$  and  $\mathcal{P}_{\omega}S / \equiv_{\mathcal{A}}$ .

First of all, let us study an important example of a finitary topology with operation. Let  $(L, \leq)$  be a distributive lattice and let  $\mathcal{I}U$ , for  $U \subseteq L$ , be the ideal generated by  $U$ . Let  $\triangleleft_{\mathcal{I}}$  be the relation between elements and finite subsets defined by  $a \triangleleft_{\mathcal{I}} K \equiv a \in \mathcal{I}K$ .

**Lemma 32.**  $(L, \triangleleft_{\mathcal{I}}, \wedge)$  is a finitary base with operation.

**Proof.** Note that  $a \triangleleft_{\mathcal{I}} \{b_1, \dots, b_n\}$  means  $a \leq b_1 \vee \dots \vee b_n$ ; use distributivity of  $L$  to prove the first rule of display (22).  $\square$

Let us write  $\triangleleft_{\mathcal{I}}$  and  $\times_{\mathcal{I}}$  for the cover relation and the binary positivity relation generated by means of the rules which are obtained by rewriting those in (21), (23) and (22) with respect to  $\triangleleft_{\mathcal{I}}$  and  $\wedge$ . Thus  $(L, \triangleleft_{\mathcal{I}}, \times_{\mathcal{I}}, \wedge)$  is a finitary formal topology with operation. Obviously, formal open subsets are just the ideals of  $L$ . Moreover,  $F$  is formal closed if and only if it satisfies

$$\frac{a \in F \quad a \leq b}{b \in F} \quad \frac{a \vee b \in F}{\{a, b\} \not\subseteq F} \quad (34)$$

for any  $a, b \in L$ . Indeed, if  $F$  satisfies the above rules, then one can use the co-induction rule

$$\frac{\begin{array}{c} [b \in P] \quad [b \in P, b \triangleleft_{\mathcal{I}} K] \\ | \quad | \\ a \in P \quad b \in V \quad K \not\subseteq P \\ \hline a \times_{\mathcal{I}} V \end{array}}{\quad} \quad (35)$$

with  $P = F = V$  and get  $a \in F \rightarrow a \times_{\mathcal{I}} F$  ( $F$  is formal closed); vice versa, the rules suggested are particular instances of the rule

$$\frac{a \triangleleft_{\mathcal{I}} K \quad a \times_{\mathcal{I}} F}{K \times_{\mathcal{I}} F} \quad (36)$$

provided that  $F$  is formal closed. Hence a formal point, that is an inhabited and convergent closed subset, is exactly a prime filter over  $(L, \leq)$ .

The following theorem says that the one above is, in fact, the only example of finitary formal topology (with operation).

**Theorem 33.** If  $\mathcal{F} = (S, \triangleleft, \times, \cdot)$  is a finitary formal topology with operation, then it is isomorphic to  $(L, \triangleleft_{\mathcal{I}}, \times_{\mathcal{I}}, \wedge)$  where  $L$  is  $K(\mathcal{A})$ , the distributive lattice of all compact subsets of  $\mathcal{F}$ .

Hence, provided that  $\mathcal{F}$  is finitary with operation,  $\text{Sat}(\mathcal{A})$  is isomorphic (as a distributive lattice) to the lattice of ideals of  $K(\mathcal{A})$ , while the space<sup>2</sup>  $\text{Pt}(\mathcal{F})$  is homeomorphic to the space of prime filters over  $K(\mathcal{A})$ .

**Proof.** Put  $L = K(\mathcal{A}) = (\mathcal{P}_{\omega}S / \equiv_{\mathcal{A}})$ , the distributive lattice of all compact subsets of  $\mathcal{F}$  (since  $\mathcal{F}$  is finitary with operation, we can identify  $L$  with the set of finite subsets of  $S$ ), and note that  $K \triangleleft_{\mathcal{I}} \{K_1, \dots, K_n\}$  is equivalent to  $K \triangleleft \bigcup_{i \leq n} K_i$ , provided that  $K \in L$  and  $\{K_1, \dots, K_n\} \subseteq_{\omega} L$ . Moreover, since  $K$  and  $\mathcal{A}K$  are equal in  $L$ , then  $\mathcal{I}\mathcal{A}K = \mathcal{I}K$ , where  $\mathcal{I}$  is the saturation operator corresponding to  $\triangleleft_{\mathcal{I}}$  (that is  $\mathcal{I}$  is the operator that generates ideals of  $L$ ). Let us consider the relations  $s$  and  $s'$  between  $S$  and  $L$  defined by

$$a s K \equiv a \triangleleft K \quad K s' a \equiv K \triangleleft \{a\} \quad (37)$$

for any  $a \in S$  and  $K \in L$  ( $s$  and  $s'$  are well defined because their definitions do not depend on the subset  $K$  chosen as representative for  $\mathcal{A}K$ ). Note that  $s^{-}K = \mathcal{A}K \equiv_{\mathcal{A}} K$  and  $(s')^{-}a = \mathcal{I}\{a\} \equiv_{\mathcal{I}} \{a\}$ . Moreover,  $(s's)^{-}a = s^{-}(s')^{-}a = s^{-}\mathcal{I}\{a\}$  is equal to  $s^{-}\{K : K \triangleleft \{a\}\} = \bigcup_{K \triangleleft \{a\}} s^{-}K = \bigcup_{K \triangleleft \{a\}} \mathcal{A}K$  which is equal (in  $\text{Sat}(\mathcal{A})$ ) to  $\{a\}$ . Similarly,  $(ss')^{-}K =_{\mathcal{I}} K$ . Now, it is easy to check that  $s$  and  $s'$  define the desired isomorphism (in the category of finitary formal topologies with operation).  $\square$

The following corollary shows that every distributive lattice is isomorphic to the lattice of compact formal open subsets of a formal topology. Thus this is a constructive version of the well known Stone's representation theorem for distributive lattices.

<sup>2</sup> See discussion after Corollary 34.

**Corollary 34** (Stone's Representation for Distributive Lattices). *Let  $L$  be a distributive lattice and let  $\mathcal{I}$  be the operator on  $\mathcal{P}L$  generating ideals. Then  $L$  is isomorphic (as a lattice) to  $K(\mathcal{I})$ , where  $K(\mathcal{I})$  is the lattice of compact elements of the formal topology  $(L, \triangleleft_{\mathcal{I}}, \times_{\mathcal{I}}, \wedge)$ .*

**Proof.** Since  $(L, \triangleleft_{\mathcal{I}}, \wedge)$  is finitary with operation, the compact elements are precisely the finite subsets modulo  $=_{\mathcal{I}}$ , that is the finitely generated ideals. But  $\mathcal{I}(\{a_1, \dots, a_n\}) = \mathcal{I}(a_1 \vee \dots \vee a_n)$ , so  $K(\mathcal{I})$  can be identified with  $\{\mathcal{I}a : a \in L\}$ . Now it is easy to see that the map:  $a \mapsto \mathcal{I}a$  is an isomorphism between distributive lattices.  $\square$

In order to justify the name we gave to the theorem above, let us see how the standard Stone's representation theorem for distributive lattices can be derived from it, provided that classical principles are used. Let  $Pt(\mathcal{I})$  be the collection of all formal points with respect to the formal topology  $(L, \triangleleft_{\mathcal{I}}, \times_{\mathcal{I}}, \wedge)$ , that is  $Pt(\mathcal{I})$  is the collection of all prime filters over  $L$ .  $Pt(\mathcal{I})$  is a topological space with

$$\{ \{F \in Pt(\mathcal{I}) : a \in F\} : a \in L \} \quad (38)$$

as a basis for the topology; moreover, that basis is closed under binary intersection because

$$(a \in F) \& (b \in F) \leftrightarrow (a \wedge b) \in F \quad (39)$$

( $F$  is a filter) and under finite unions because

$$\{F \in Pt(\mathcal{I}) : a \in F\} \cup \{F \in Pt(\mathcal{I}) : b \in F\} = \{F \in Pt(\mathcal{I}) : a \vee b \in F\} \quad (40)$$

( $F$  is prime). It is easy to check that  $\{a\} \subseteq L$  is compact as an element of  $Sat(\mathcal{I})$  if and only if  $\{F \in Pt(\mathcal{I}) : a \in F\}$  is a compact open subset (in the usual sense) in the topology on  $Pt(\mathcal{I})$ . Moreover, if  $\mathcal{U} \subseteq Pt(\mathcal{I})$  is a compact open subset then  $\mathcal{U}$  is the union of a finite number of elements of the basis and thus belongs to it. Finally, the map

$$a \mapsto \{F \in Pt(\mathcal{I}) : a \in F\} \quad (41)$$

is a lattice isomorphism between  $L$  and the compact open subsets of the topology on  $Pt(\mathcal{I})$ .

We now want to investigate the link between finitary formal topologies and coherent locales (in an impredicative framework, every locale is of the kind  $Sat(\mathcal{A})$  for some  $\mathcal{A}$ ; see [1]).

**Definition 35.** Let  $\mathcal{A}$  be a saturation operator on a set  $S$  and let  $K(\mathcal{A})$  be the collection of all compact elements in  $Sat(\mathcal{A})$ . We say that  $Sat(\mathcal{A})$  is *coherent* (or *compactly based*) if the following hold:

0.  $K(\mathcal{A})$  is (at least)<sup>3</sup> a setoid;
1.  $K(\mathcal{A})$  is a sub-lattice of  $Sat(\mathcal{A})$ ;
2. each element of  $Sat(\mathcal{A})$  can be obtained as a possibly infinite join of elements in  $K(\mathcal{A})$ .

**Theorem 36.** *If  $\mathcal{S} = (S, \triangleleft, \cdot)$  is a finitary cover with operation, then  $Sat(\mathcal{A})$  is coherent.*

**Proof.**  $\mathcal{S}$  is finitary, so  $K(\mathcal{A})$  can be identified with  $\mathcal{P}_{\omega}S$  and then with  $List(S)$ . We already know that  $K(\mathcal{A})$  is a sub-lattice of  $Sat(\mathcal{A})$ , since  $\triangleleft$  is finitary. Moreover, we have:

$$\mathcal{A}V = \mathcal{A} \bigcup_{a \in V} \{a\} = \bigvee_{a \in V} \mathcal{A}\{a\}, \quad (42)$$

so  $V =_{\mathcal{A}} \bigvee_{a \in V} \mathcal{A}\{a\}$  where each  $\mathcal{A}\{a\}$  is compact because  $\{a\}$  is finite and  $\triangleleft$  is finitary.  $\square$

Note that the converse of the previous theorem does not hold, that is  $Sat(\mathcal{A})$  can be coherent without  $\mathcal{S}$  being finitary. For instance, let  $S = \mathbb{N} \cup \{*\}$  and let  $\triangleleft$  be the smallest cover relation on  $S$  which satisfies  $* \triangleleft \mathbb{N}$  and  $n \triangleleft \{*\}$ , for any  $n \in \mathbb{N}$ . Note that the restriction of  $\triangleleft$  to  $\mathbb{N} \times \mathcal{P}\mathbb{N}$  is just membership; then  $K(\mathcal{A})$  is exactly  $\mathcal{P}_{\omega}\mathbb{N}$  (a subset containing  $*$  is surely not compact, since  $*$  is covered by  $\mathbb{N}$  and by none of its finite subsets), hence a setoid. Moreover,  $\mathcal{P}_{\omega}\mathbb{N}$  is a lattice, because the intersection of two finite subsets of  $\mathbb{N}$  is finite too (the equality in  $\mathbb{N}$  is decidable). These facts can be used to prove that  $Sat(\mathcal{A})$  is coherent. But  $(S, \triangleleft)$  is not finitary because the singleton  $\{*\}$  is not compact.

In other words, being coherent is a property of locales and not of their bases; however, we can show that coherent locales can always be presented via finitary topologies.

**Theorem 37.** *Let  $\mathcal{S} = (S, \triangleleft)$  be a cover such that  $Sat(\mathcal{A})$  is coherent. Then  $\mathcal{S}$  is isomorphic (as a cover) to a finitary cover with operation  $\mathcal{T} = (T, \triangleleft', \cdot)$ , where  $T = K(\mathcal{A})$ , the setoid of compact elements of  $Sat(\mathcal{A})$ ,  $\cdot$  is the  $\wedge$  of  $K(\mathcal{A})$  and  $V \triangleleft' \mathcal{U} \equiv V \triangleleft \bigcup_{U \in \mathcal{U}} U$  for  $V \in T$  and  $\mathcal{U} \subseteq T$ .*

**Proof.** It is easily proved that  $\triangleleft'$  is a cover with operation, that is  $\triangleleft'$  is a cover and satisfies  $\cdot$ —Right and  $\cdot$ —Left of Definition 4. Moreover,  $\triangleleft'$  is finitary because  $T = K(\mathcal{A})$ . Note also that  $V \triangleleft' \{U\}$  if and only if  $V \triangleleft U$ , provided that  $V, U \in T$ .

For  $a \in S$  and  $V \in T$ , put  $asV \equiv a \triangleleft V$  and  $Vs'a \equiv V \triangleleft a$ . For any  $a \in S$  there exists a set  $I(a)$  and a family  $\{V_i\}_{i \in I(a)} \subseteq K(\mathcal{A})$  such that  $a \triangleleft \bigcup_{i \in I(a)} V_i$  and  $V_i \triangleleft a$  for any  $i \in I(a)$ , because  $Sat(\mathcal{A})$  is coherent. Thus  $\{V_i\}_{i \in I(a)} \subseteq (s')^{-}a$ , then  $a \triangleleft (s')^{-}a$  and  $a =_{\mathcal{A}} (s')^{-}a$ . Similarly  $U =_{\mathcal{A}} (s')^{-}U$ , for any  $U \subseteq S$ . Moreover,  $(s's)^{-}a = \bigcup_{V \triangleleft a} \mathcal{A}V$  is equal to  $\{a\}$  modulo  $=_{\mathcal{A}}$  and  $(ss')^{-}V = \bigcup_{W \triangleleft V} (s')^{-}W$  is equal to  $\{V\}$  modulo  $=_{\mathcal{A}'}$ . Now it is easy to check that  $s$  and  $s'$  define the desired isomorphism between covers. Note that  $s^{-*}$  and  $(s')^{-*}$  define a bijective correspondence between  $K(\mathcal{A})$  and  $\mathcal{P}_{\omega}(T)/=_{\mathcal{A}'}$ .  $\square$

<sup>3</sup> In other words,  $K(\mathcal{A})$  must either be a set or a quotient set. Thus  $K(\mathcal{A})$  cannot be a proper collection like  $\mathcal{P}S$ . Equivalently,  $K(\mathcal{A})$  could be indexed by a set, because in that case it could be identified with a suitable quotient set of the index set.

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